

ON THE COUPLING PROPERTY AND THE LIOUVILLE THEOREM FOR ORNSTEIN-UHLENBECK PROCESSES

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ABSTRACT. Using a coupling for the weighted sum of independent random variables and the explicit expression of the transition semigroup of Ornstein-Uhlenbeck processes driven by compound Poisson processes, we establish the existence of a successful coupling and the Liouville theorem for general Ornstein-Uhlenbeck processes. Then we present the explicit coupling property of Ornstein-Uhlenbeck processes directly from the behaviour of the corresponding symbol or characteristic exponent. This approach allows us to derive gradient estimates for Ornstein-Uhlenbeck processes via the symbol.

Keywords: Ornstein-Uhlenbeck processes; coupling property; Liouville theorem; gradient estimates.

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1. MAIN RESULTS

Let $(X_t^x)_{t \geq 0}$ be an n -dimensional Ornstein-Uhlenbeck process, which is defined as the unique strong solution of the following stochastic differential equation

$$(1.1) \quad dX_t = AX_t dt + B dZ_t, \quad X_0 = x \in \mathbb{R}^n.$$

Here A is a real $n \times n$ matrix, B is a real $n \times d$ matrix and Z_t is a Lévy process in \mathbb{R}^d ; note that we allow Z_t to take values in a proper subspace of \mathbb{R}^d . It is well known that

$$X_t^x = e^{tA}x + \int_0^t e^{(t-s)A} B dZ_s.$$

The characteristic exponent or symbol Φ of Z_t , defined by

$$\mathbb{E}(e^{i\langle \xi, Z_t \rangle}) = e^{-t\Phi(\xi)}, \quad \xi \in \mathbb{R}^d,$$

enjoys the following Lévy-Khintchine representation:

$$(1.2) \quad \Phi(\xi) = \frac{1}{2} \langle Q\xi, \xi \rangle + i \langle b, \xi \rangle + \int_{z \neq 0} \left(1 - e^{i\langle \xi, z \rangle} + i \langle \xi, z \rangle \mathbf{1}_{B(0,1)}(z) \right) \nu(dz),$$

where $Q = (q_{j,k})_{j,k=1}^d$ is a positive semi-definite matrix, $b \in \mathbb{R}^d$ is the drift vector and ν is the Lévy measure, i.e. a σ -finite measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{z \neq 0} (1 \wedge |z|^2) \nu(dz) < \infty$.

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For every $\varepsilon > 0$, define ν_ε on \mathbb{R}^d as follows:

$$\nu_\varepsilon(C) = \begin{cases} \nu(C), & \text{if } \nu(\mathbb{R}^d) < \infty; \\ \nu(C \setminus \{z : |z| < \varepsilon\}), & \text{if } \nu(\mathbb{R}^d) = \infty. \end{cases}$$

Let $(Y_t)_{t \geq 0}$ be a Markov process on \mathbb{R}^n with transition function $P_t(x, \cdot)$. Then, according to [5, 15, 13], we say that $(Y_t)_{t \geq 0}$ admits a *successful coupling* (also: enjoys the *coupling property*) if for any $x, y \in \mathbb{R}^n$,

$$\lim_{t \rightarrow \infty} \|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{Var}} = 0,$$

where $\|\cdot\|_{\text{Var}}$ stands for the total variation norm. If a Markov process admits a successful coupling, then it also has the Liouville property, i.e. every bounded harmonic function is constant; in this context a function f is harmonic, if $Lf = 0$ where L is the generator of the Markov process. See [3, 4] and the references therein for this result and more details on the coupling property.

Let A be an $n \times n$ matrix. We say that an eigenvalue λ of A is *semisimple* if the dimension of the corresponding eigenspace is equal to the algebraic multiplicity of λ as a root of characteristic polynomial of A . Note that for symmetric matrices A all eigenvalues are real and semisimple. Recall that for any two bounded measures μ and ν on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, $\mu \wedge \nu := \mu - (\mu - \nu)^+$, where $(\mu - \nu)^\pm$ refers to the Jordan-Hahn decomposition of the signed measure $\mu - \nu$. In particular, $\mu \wedge \nu = \nu \wedge \mu$, and $\mu \wedge \nu(\mathbb{R}^d) = \frac{1}{2}[\mu(\mathbb{R}^d) + \nu(\mathbb{R}^d) - \|\mu - \nu\|_{\text{Var}}]$.

One of our main results is the following

Theorem 1.1. *Let $P_t(x, \cdot)$ be the transition probability of the Ornstein-Uhlenbeck process $\{X_t^x\}_{t \geq 0}$ given by (1.1). Assume that $\text{Rank}(B) = n$ (which implies $n \leq d$), and that there exist $\varepsilon, \delta > 0$ such that*

$$(1.3) \quad \inf_{z \in \mathbb{R}^d, |z| \leq \delta} \nu_\varepsilon \wedge (\delta_z * \nu_\varepsilon)(\mathbb{R}^d) > 0.$$

If the real parts of all eigenvalues of A are non-positive and if all purely imaginary eigenvalues are semisimple, then there exists a constant $C = C(\varepsilon, \delta, \nu, A, B) > 0$ such that for all $x, y \in \mathbb{R}^n$ and $t > 0$,

$$(1.4) \quad \|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{Var}} \leq \frac{C(1 + |x - y|)}{\sqrt{t}} \wedge 2.$$

As a consequence of Theorem 1.1, we immediately obtain the following result which partly answers the following question about Liouville theorems for non-local operators from [7, page 458]: *A challenging task would be to apply other probabilistic techniques, based on ... coupling to non-local operators.*

Corollary 1.2. *Under the conditions of Theorem 1.1, the Ornstein-Uhlenbeck process $\{X_t^x\}_{t \geq 0}$ admits a successful coupling and has the Liouville property.*

Remark 1.3 (The conditions of Theorem 1.1 are optimal). (1) If $A = 0$, $d = n$ and $B = \text{id}_{\mathbb{R}^n}$, then X_t is just a Lévy process on \mathbb{R}^n . The condition (1.3) is one possibility to guarantee sufficient jump activity such that the Lévy process X_t admits a successful coupling. To see that (1.3) is sharp, we can use the example in [13, Remark 1.2].

(2) Let Z_t be a (rotationally symmetric) α -stable Lévy process Z_t , $0 < \alpha < 2$, and denote by X_t the n -dimensional Ornstein-Uhlenbeck process driven by Z_t , i.e.

$$dX_t = AX_t dt + dZ_t.$$

If at least one eigenvalue of A has positive real part, then X_t does not have the coupling property. Indeed, according to [7, Example 3.4 and Theorem 3.5], we know that X_t does not have the Liouville property, i.e. there exists a bounded harmonic function which is not constant. According to [5, Theorem 21.12] or [3, Theorem 1 and its second remark], X_t does not have the coupling property. This example indicates that the non-positivity of the real parts of the eigenvalues of A is also necessary.

Remark 1.4 (Strong Feller property vs. coupling property). In [13, Theorem 4.1 and Corollary 4.2] we show that Lévy processes which have the strong Feller property admit the coupling property. A similar conclusion, however, does not hold for general Ornstein-Uhlenbeck processes. Consider, for instance, the one-dimensional Ornstein-Uhlenbeck process given by

$$dX_t = X_t dt + dZ_t, \quad X_0 = x \in \mathbb{R},$$

where Z_t is an α -stable Lévy process Z_t on \mathbb{R} . According to [8, Theorem 1.1] (or [6, Theorem A]) and [8, Proposition 2.1], we know that X_t has the strong Feller property. However, the argument used in Remark 1.3 shows that this process fails to have the coupling property.

Recently, F.-Y. Wang [16] has studied the coupling property of an Ornstein-Uhlenbeck process X_t defined by (1.1). Assume that $\text{Rank}(B) = n$ and $\langle Ax, x \rangle \leq 0$ holds for $x \in \mathbb{R}^n$. In [16, Theorem 3.1] it is proved that (1.4) is satisfied for some constant $C > 0$, whenever the Lévy measure of Z_t satisfies $\nu(dz) \geq \rho_0(z)dz$ such that

$$(1.5) \quad \int_{\{|z-z_0| \leq \varepsilon\}} \frac{dz}{\rho_0(z)} < \infty$$

holds for some $z_0 \in \mathbb{R}^d$ and some $\varepsilon > 0$.

Let us compare F.-Y. Wang's result with our Theorem 1.1.

Proposition 1.5. *Assume that (1.5) holds for some $\rho_0 \in L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$, some $z_0 \in \mathbb{R}^d$ and some $\varepsilon > 0$. Then, there exist a closed subset $F \subset \overline{B}(z_0, \varepsilon) = \{z \in \mathbb{R}^d : |z - z_0| \leq \varepsilon\}$ and a constant $\delta > 0$ such that*

$$\inf_{x \in \mathbb{R}^d, |x| \leq \delta} \int_F (\rho_0(z) \wedge \rho_0(z - x)) dz > 0.$$

We postpone the technical proof of Proposition 1.5 to Section 3.2 in the appendix. Proposition 1.5 shows that Theorem 1.1 improves [16, Theorem 3.1], even if the Lévy measure ν of Z_t has an absolutely continuous component as we will see in the following example.

Example 1.6. Let $C_{3/4}$ be a Smith-Volterra-Cantor set in $[0, 1]$ with Lebesgue measure $\text{Leb}(C_{3/4}) = 3/4$, i.e. $C_{3/4}$ is a perfect set with empty interior, see e.g. [1, Chapter 3, Section 18]. Consider the following one-dimensional Ornstein-Uhlenbeck process

$$dX_t = -X_t dt + dZ_t, \quad X_0 = x \in \mathbb{R},$$

where Z_t is a real-valued Lévy process with Lévy measure $\nu(dz) = \mathbf{1}_{C_{3/4}}(z) dz$. We will see that we can use Theorem 1.1 to show the coupling property of the process X_t while the criterion from [16, Theorem 3.1] fails.

Let $\delta \in (0, 1/8)$ and $z \in [-\delta, \delta]$. Then

$$\begin{aligned} \nu_\varepsilon \wedge (\delta_z * \nu_\varepsilon)(\mathbb{R}) &= \int \left(\mathbf{1}_{C_{3/4}}(x) \wedge \mathbf{1}_{C_{3/4}}(x+z) \right) dx \\ &= \text{Leb}(C_{3/4} \cap (C_{3/4} - z)) \\ &= \text{Leb}(C_{3/4}) + \text{Leb}(C_{3/4} - z) - \text{Leb}(C_{3/4} \cup (C_{3/4} - z)) \\ &\geq \frac{6}{4} - \text{Leb}[-|z|, 1+|z|] \geq \frac{1}{4}. \end{aligned}$$

This shows that the conditions of Theorem 1.1 are satisfied.

On the other hand, since $C_{3/4}$ contains no intervals, we see that for all $z_0 \in \mathbb{R}$ and $\varepsilon > 0$,

$$\int_{\{|z-z_0| \leq \varepsilon\}} \frac{dz}{\mathbf{1}_{C_{3/4}}(z)} = \infty$$

(here we use the convention $\frac{1}{0} = +\infty$). This means that (1.5) does not hold.

Now we are going to estimate $\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}}$ for large values of t with the help of the characteristic exponent $\Phi(\xi)$ of the Lévy process Z_t . We restrict ourselves to the case where $Q = 0$ in (1.2), i.e. to Lévy process $(Z_t)_{t \geq 0}$ without a Gaussian part. For $t, \rho > 0$, define

$$\varphi_t(\rho) := \sup_{|\xi| \leq \rho} \int_0^t \text{Re} \Phi(B^\top e^{sA^\top} \xi) ds,$$

where M^\top denotes the transpose of the matrix M .

Theorem 1.7. *Let $P_t(x, \cdot)$ be the transition function of the Ornstein-Uhlenbeck process $\{X_t^x\}_{t \geq 0}$ on \mathbb{R}^n given by (1.1). Assume that there exists some $t_0 > 0$ such that*

$$(1.6) \quad \liminf_{|\xi| \rightarrow \infty} \frac{\int_0^{t_0} \text{Re} \Phi(B^\top e^{sA^\top} \xi) ds}{\log(1 + |\xi|)} > 2n + 2.$$

If

$$(1.7) \quad \int \exp \left(- \int_0^t \text{Re} \Phi(B^\top e^{sA^\top} \xi) ds \right) |\xi|^{n+2} d\xi = O(\varphi_t^{-1}(1)^{2n+2}) \quad \text{as } t \rightarrow \infty,$$

then there exist $t_1, C > 0$ such that for any $x, y \in \mathbb{R}^n$ and $t \geq t_1$,

$$(1.8) \quad \|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} \leq C |e^{tA}(x - y)| \varphi_t^{-1}(1).$$

In particular, when

$$(1.9) \quad \xi \mapsto \int_0^\infty \text{Re} \Phi(B^\top e^{sA^\top} \xi) ds \quad \text{is locally bounded,}$$

we only need the condition (1.6) to get (1.8).

Note that (1.9) is, e.g. satisfied, if the real parts of all eigenvalues of A are negative and

$$\limsup_{|\xi| \rightarrow 0} \frac{\operatorname{Re} \Phi(B^\top \xi)}{|\xi|^\kappa} < \infty$$

for some constant $\kappa > 0$.

The remaining part of this paper is organized as follows. In Section 2 we first present the proof of Theorem 1.1, where a coupling for the weighted sum of independent random variables and the explicit expression of the transition semigroup of Ornstein-Uhlenbeck processes driven by a compound Poisson process are used. Then, we follow the approach of our recent paper [12] to prove Theorem 1.7. As a byproduct, we also derive explicit gradient estimates for Ornstein-Uhlenbeck processes, cf. the Appendix 3.1.

2. PROOFS OF THEOREMS

We begin with the proof of Theorem 1.1.

Proof of Theorem 1.1. The proof is split into six steps.

Step 1. For any $\varepsilon > 0$, let $(Z_t^\varepsilon)_{t \geq 0}$ be a compound Poisson process on \mathbb{R}^d whose Lévy measure is ν_ε . Then, $(Z_t^\varepsilon)_{t \geq 0}$ and $(Z_t - Z_t^\varepsilon)_{t \geq 0}$ are independent Lévy processes. It follows, in particular, that the random variables

$$X_t^{\varepsilon, x} := e^{tA}x + \int_0^t e^{(t-s)A} B dZ_s^\varepsilon$$

and

$$X_t^x - X_t^{\varepsilon, x} := \int_0^t e^{(t-s)A} B d(Z_s - Z_s^\varepsilon)$$

are independent for any $\varepsilon > 0$ and $t \geq 0$.

Step 2. Denote by $\mu_{\varepsilon, t}$ the law of random variable

$$X_t^{\varepsilon, 0} := X_t^{\varepsilon, x} - e^{tA}x = \int_0^t e^{(t-s)A} B dZ_s^\varepsilon.$$

We will compute $\mu_{\varepsilon, t}$, which coincides with the law of $\int_0^t e^{sA} B dZ_s^\varepsilon$, cf. Lemma 2.2 below. Our argument follows the proof of [8, Theorem 1.1], which is motivated by [10, Theorem 27.7].

The law of the compound poisson process Z_t^ε is given by

$$e^{-C_\varepsilon t} \left[\delta_0 + \sum_{k=1}^{\infty} \frac{(C_\varepsilon t)^k}{k!} \bar{\nu}_\varepsilon^{*k} \right],$$

where $C_\varepsilon = \nu_\varepsilon(\mathbb{R}^d)$, $\bar{\nu}_\varepsilon = \nu_\varepsilon / C_\varepsilon$ and $\bar{\nu}_\varepsilon^{*k}$ is the k -fold convolution of $\bar{\nu}_\varepsilon$.

Construct a sequence $(\xi_i)_{i \geq 1}$ of iid random variables which are exponentially distributed with intensity C_ε , and introduce a further sequence $(U_i)_{i \geq 1}$ of iid random

variables on \mathbb{R}^d with law $\bar{\nu}_\varepsilon$. We will assume that the random variables $(U_i)_{i \geq 1}$ are independent of the sequence $(\xi_i)_{i \geq 1}$. It is not difficult to check that the random variable

$$(2.10) \quad 0 \cdot \mathbb{1}_{\{\xi_1 > t\}} + \sum_{k=1}^{\infty} \mathbb{1}_{\{\xi_1 + \dots + \xi_k \leq t < \xi_1 + \dots + \xi_{k+1}\}} \left(e^{\xi_1 A} B U_1 + \dots + e^{(\xi_1 + \dots + \xi_k) A} B U_k \right)$$

also has the probability distribution $\mu_{\varepsilon, t}$.

Using (2.10) we find for any $f \in B_b(\mathbb{R}^n)$,

$$(2.11) \quad \mathbb{E}f(X_t^{\varepsilon, x}) = \int f(e^{tA}x + z) \mu_{\varepsilon, t}(dz) = f(e^{tA}x) e^{-C_\varepsilon t} + Hf(x),$$

where

$$\begin{aligned} Hf(x) &:= \mathbb{E}f \left(\sum_{k=1}^{\infty} \mathbb{1}_{\{\xi_1 + \dots + \xi_k \leq t < \xi_1 + \dots + \xi_{k+1}\}} \left(e^{tA}x + e^{\xi_1 A} B U_1 + \dots + e^{(\xi_1 + \dots + \xi_k) A} B U_k \right) \right) \\ &= \sum_{k=1}^{\infty} \mathbb{E}f \left(\mathbb{1}_{\{\xi_1 + \dots + \xi_k \leq t < \xi_1 + \dots + \xi_{k+1}\}} \left(e^{tA}x + e^{\xi_1 A} B U_1 + \dots + e^{(\xi_1 + \dots + \xi_k) A} B U_k \right) \right) \\ &= \sum_{k=1}^{\infty} \int \dots \int_{t_1 + \dots + t_k \leq t < t_1 + \dots + t_{k+1}} C_\varepsilon^{k+1} e^{-C_\varepsilon(t_1 + \dots + t_{k+1})} dt_1 \dots dt_{k+1} \times \\ &\quad \times \int \dots \int_{\mathbb{R}^d} f(e^{tA}x + e^{t_1 A} B y_1 + \dots + e^{(t_1 + \dots + t_k) A} B y_k) \bar{\nu}_\varepsilon(dy_1) \dots \bar{\nu}_\varepsilon(dy_k) \\ &= \sum_{k=1}^{\infty} \int \dots \int_{t_1 + \dots + t_k \leq t < t_1 + \dots + t_{k+1}} C_\varepsilon^{k+1} e^{-C_\varepsilon(t_1 + \dots + t_{k+1})} dt_1 \dots dt_{k+1} \times \\ &\quad \times \int_{\mathbb{R}^n} f(e^{tA}x + z) \mu_{t_1, \dots, t_k}(dz). \end{aligned}$$

Here μ_{t_1, \dots, t_k} is the probability measure on \mathbb{R}^n which is the image of the k -fold product measure $\bar{\nu}_\varepsilon \times \dots \times \bar{\nu}_\varepsilon$ under the linear transformation J_{t_1, \dots, t_k} (independent of ε) acting from $(\mathbb{R}^d)^k$ into \mathbb{R}^n :

$$J_{t_1, \dots, t_k}(y_1, \dots, y_k) = e^{t_1 A} B y_1 + \dots + e^{(t_1 + \dots + t_k) A} B y_k,$$

for $y_i \in \mathbb{R}^d$ and $i = 1, \dots, k$.

Step 3. Let $P_t(x, \cdot)$ and P_t be the transition function and the transition semigroup of the Ornstein-Uhlenbeck process $(X_t^x)_{t \geq 0}$. Similarly, we denote by $P_t^\varepsilon(x, \cdot)$ and P_t^ε the transition function and the transition semigroup of $(X_t^{\varepsilon, x})_{t \geq 0}$, and by $Q_t^\varepsilon(x, \cdot)$ and Q_t^ε the transition function and the transition semigroup of $(X_t^x - X_t^{\varepsilon, x})_{t \geq 0}$. By the

independence of the processes $(X_t^{\varepsilon,x})_{t \geq 0}$ and $(X_t^x - X_t^{\varepsilon,x})_{t \geq 0}$, we get

$$\begin{aligned}
 \|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{Var}} &= \sup_{\|f\|_{\infty} \leq 1} |P_t f(x) - P_t f(y)| \\
 &= \sup_{\|f\|_{\infty} \leq 1} |P_t^{\varepsilon} Q_t^{\varepsilon} f(x) - P_t^{\varepsilon} Q_t^{\varepsilon} f(y)| \\
 &\leq \sup_{\|h\|_{\infty} \leq 1} |P_t^{\varepsilon} h(x) - P_t^{\varepsilon} h(y)|.
 \end{aligned}
 \tag{2.12}$$

Furthermore, it follows from (2.11) that

$$\begin{aligned}
 &\sup_{\|h\|_{\infty} \leq 1} |P_t^{\varepsilon} h(x) - P_t^{\varepsilon} h(y)| \\
 &\leq 2e^{-C_{\varepsilon}t} + \sum_{k=1}^{\infty} \int_{t_1+\dots+t_k \leq t < t_1+\dots+t_{k+1}} C_{\varepsilon}^{k+1} e^{-C_{\varepsilon}(t_1+\dots+t_{k+1})} dt_1 \dots dt_{k+1} \times \\
 &\quad \times \sup_{\|h\|_{\infty} \leq 1} \left| \int_{\mathbb{R}^n} h(e^{tA}x + z) \mu_{t_1, \dots, t_k}(dz) - \int_{\mathbb{R}^n} h(e^{tA}y + z) \mu_{t_1, \dots, t_k}(dz) \right| \\
 &= 2e^{-C_{\varepsilon}t} + \sum_{k=1}^{\infty} \int_{t_1+\dots+t_k \leq t < t_1+\dots+t_{k+1}} C_{\varepsilon}^{k+1} e^{-C_{\varepsilon}(t_1+\dots+t_{k+1})} dt_1 \dots dt_{k+1} \times \\
 &\quad \times \sup_{\|h\|_{\infty} \leq 1} \left| \int_{\mathbb{R}^n} h(e^{tA}(x-y) + z) \mu_{t_1, \dots, t_k}(dz) - \int_{\mathbb{R}^n} h(z) \mu_{t_1, \dots, t_k}(dz) \right| \\
 &\leq 2e^{-C_{\varepsilon}t} + \sum_{k=1}^{\infty} \int_{t_1+\dots+t_k \leq t < t_1+\dots+t_{k+1}} C_{\varepsilon}^{k+1} e^{-C_{\varepsilon}(t_1+\dots+t_{k+1})} dt_1 \dots dt_{k+1} \times \\
 &\quad \times \|\delta_{e^{tA}(x-y)} * \mu_{t_1, \dots, t_k} - \mu_{t_1, \dots, t_k}\|_{\text{Var}}.
 \end{aligned}
 \tag{2.13}$$

Step 4. For any $a \in \mathbb{R}^n$, $a \neq 0$, let R_a be the non-degenerate rotation such that $R_a a = |a|e_1$. Then, by [13, Lemma 3.2],

$$\begin{aligned}
 &\|\delta_{e^{tA}(x-y)} * \mu_{t_1, \dots, t_k} - \mu_{t_1, \dots, t_k}\|_{\text{Var}} \\
 &= \|\delta_{|e^{tA}(x-y)|e_1} * (\mu_{t_1, \dots, t_k} \circ R_{e^{tA}(x-y)}^{-1}) - \mu_{t_1, \dots, t_k} \circ R_{e^{tA}(x-y)}^{-1}\|_{\text{Var}}.
 \end{aligned}$$

Since μ_{t_1, \dots, t_k} is the law of the random variable

$$\sum_{i=1}^k e^{(t_1+\dots+t_i)A} B U_i,$$

$\mu_{t_1, \dots, t_k} \circ R_{e^{tA}(x-y)}^{-1}$ is the law of the random variable

$$\sum_{i=1}^k R_{e^{tA}(x-y)}(e^{(t_1+\dots+t_i)A} B U_i).$$

To estimate $\|\delta_{|e^{tA}(x-y)|e_1} * (\mu_{t_1, \dots, t_k} \circ R_{e^{tA}(x-y)}^{-1}) - \mu_{t_1, \dots, t_k} \circ R_{e^{tA}(x-y)}^{-1}\|_{\text{Var}}$, we will use the Mineka and Lindvall-Rogers couplings for random walks. The remainder of this

part is based on the proof of [13, Proposition 3.3]. In order to ease notations, we set $\mathbf{n} := \bar{\nu}_\varepsilon$ and $\mathbf{n}^a := \delta_a * \bar{\nu}_\varepsilon$ for any $a \in \mathbb{R}^d$.

Since $\text{Rank}(B) = n$, there exists a real $d \times n$ matrix \bar{B} such that $B\bar{B} = \text{id}_{\mathbb{R}^n}$, see e.g. [2, Theorem 2.6.1, Page 35]. For any $i \geq 1$, let $(U_i, \Delta U_i) \in \mathbb{R}^d \times \mathbb{R}^d$ be a pair of random variables with the following distribution

$$\mathbb{P}((U_i, \Delta U_i) \in C \times D) = \begin{cases} \frac{1}{2}(\mathbf{n} \wedge \mathbf{n}^{-a_i})(C), & \text{if } D = \{a_i\}; \\ \frac{1}{2}(\mathbf{n} \wedge \mathbf{n}^{a_i})(C), & \text{if } D = \{-a_i\}; \\ (\mathbf{n} - \frac{1}{2}(\mathbf{n} \wedge \mathbf{n}^{-a_i} + \mathbf{n} \wedge \mathbf{n}^{a_i}))(C), & \text{if } D = \{0\}; \end{cases}$$

where $C \in \mathcal{B}(\mathbb{R}^d)$, $a_i = \bar{B} e^{(t_1 + \dots + t_i)A} (x - y)$ and D is any of the following three sets: $\{-a_i\}$, $\{0\}$ or $\{a_i\}$. Again by [13, Lemma 3.2],

$$\begin{aligned} \mathbb{P}(\Delta U_i = -a_i) &= \frac{1}{2}(\mathbf{n} \wedge (\delta_{a_i} * \mathbf{n}))(\mathbb{R}^d) \\ &= \frac{1}{2}(\mathbf{n} \wedge (\delta_{-a_i} * \mathbf{n}))(\mathbb{R}^d) \\ &= \mathbb{P}(\Delta U_i = a_i). \end{aligned}$$

It is clear that the distribution of U_i is \mathbf{n} . Let $U'_i = U_i + \Delta U_i$. We claim that the distribution of U'_i is also \mathbf{n} . Indeed, for any $C \in \mathcal{B}(\mathbb{R}^d)$,

$$\begin{aligned} \mathbb{P}(U'_i \in C) &= \mathbb{P}(U_i - a_i \in C, \Delta U_i = -a_i) + \mathbb{P}(U_i + a_i \in C, \Delta U_i = a_i) + \mathbb{P}(U_i \in C, \Delta U_i = 0) \\ &= \frac{1}{2}(\delta_{-a_i} * (\mathbf{n} \wedge \mathbf{n}^{a_i}))(C) + \frac{1}{2}(\delta_{a_i} * (\mathbf{n} \wedge \mathbf{n}^{-a_i}))(C) + \left(\mathbf{n} - \frac{1}{2}(\mathbf{n} \wedge \mathbf{n}^{-a_i} + \mathbf{n} \wedge \mathbf{n}^{a_i}) \right)(C) \\ &= \mathbf{n}(C), \end{aligned}$$

where we have used that

$$\delta_{a_i} * (\mathbf{n} \wedge \mathbf{n}^{-a_i}) = \mathbf{n} \wedge \mathbf{n}^{a_i} \quad \text{and} \quad \delta_{-a_i} * (\mathbf{n} \wedge \mathbf{n}^{a_i}) = \mathbf{n} \wedge \mathbf{n}^{-a_i}.$$

Without loss of generality, we can assume that the pairs (U_i, U'_i) are independent for all $i \geq 1$. Now we construct the coupling

$$(S_k, S'_k)_{k \geq 1} = \left(\sum_{i=1}^k R_{e^{tA}(x-y)}(e^{(t_1 + \dots + t_i)A} B U_i), \sum_{i=1}^k R_{e^{tA}(x-y)}(e^{(t_1 + \dots + t_i)A} B U'_i) \right)_{k \geq 1}$$

of

$$S_k := \sum_{i=1}^k R_{e^{tA}(x-y)}(e^{(t_1 + \dots + t_i)A} B U_i).$$

Since $U'_i - U_i = \Delta U_i$ is either $\pm a_i$ or 0, we know that

$$\begin{aligned} (S_k - S'_k)_{k \geq 1} &= \left(\sum_{i=1}^k R_{e^{tA}(x-y)}(e^{(t_1+\dots+t_i)A} B U'_i) - \sum_{i=1}^k R_{e^{tA}(x-y)}(e^{(t_1+\dots+t_i)A} B U_i) \right)_{k \geq 1} \\ &= \left(\sum_{i=1}^k R_{e^{tA}(x-y)}(e^{(t_1+\dots+t_i)A} B (U'_i - U_i)) \right)_{k \geq 1} \end{aligned}$$

is a random walk on \mathbb{R}^n whose steps are symmetrically (but not necessarily identically) distributed and take only the values $\pm |e^{tA}(x-y)|e_1$ and 0.

Set $S_k^j = \sum_{i=1}^k \eta_i^j$ and $S_k^{j'} = \sum_{i=1}^k \eta_i^{j'}$ for $1 \leq j \leq n$, where

$$(\eta_i^1, \dots, \eta_i^n) = R_{e^{tA}(x-y)}(e^{(t_1+\dots+t_i)A} B U_i)$$

and

$$(\eta_i^{1'}, \dots, \eta_i^{n'}) = R_{e^{tA}(x-y)}(e^{(t_1+\dots+t_i)A} B U'_i).$$

Then $(S_k^1 - S_k^{1'})_{k \geq 1}$ is a random walk on \mathbb{R} whose steps are independent and attain the values $-|e^{tA}(x-y)|$, 0 and $|e^{tA}(x-y)|$ with probabilities $\frac{1}{2}(1-p_i)$, p_i and $\frac{1}{2}(1-p_i)$, respectively; the values of the p_i are given by

$$\begin{aligned} p_i &:= \mathbb{P}(\eta_i^{1'} - \eta_i^1 = 0) \\ &= (\mathbf{n} - \tfrac{1}{2}(\mathbf{n} \wedge \mathbf{n}^{-a_i} + \mathbf{n} \wedge \mathbf{n}^{a_i}))(\mathbb{R}^d) \\ &= 1 - \mathbf{n} \wedge \mathbf{n}^{-a_i}(\mathbb{R}^d). \end{aligned}$$

Since $S_k^j = S_k^{j'}$ for $2 \leq j \leq n$, we get

$$(2.14) \quad \|\delta_{e^{tA}(x-y)} * \mu_{t_1, \dots, t_k} - \mu_{t_1, \dots, t_k}\|_{\text{Var}} \leq 2 \mathbb{P}(T^S > k),$$

where

$$T^S = \inf\{i \geq 1 : S_i^1 = S_i^{1'} + |e^{tA}(x-y)|\}.$$

Step 5. Since the real parts of all eigenvalues of A are non-positive and since all purely imaginary eigenvalues are semisimple, we know from [2, Proposition 11.7.2, Page 438] that $C_A := \sup_{t \geq 0} \|e^{tA}\| < \infty$. In particular, when $t \geq t_1 + \dots + t_i$,

$$|e^{(t-(t_1+\dots+t_i))A}(x-y)| \leq C_A |x-y|.$$

From (1.3) we get that for all $i \geq 1$ and $x, y \in \mathbb{R}^n$ with $|x-y| \leq \delta(C_A \|\bar{B}\|)^{-1}$,

$$\begin{aligned} \frac{1}{2}(1-p_i) &= \frac{1}{2}(\mathbf{n} \wedge (\delta_{-a_i} * \mathbf{n}))(\mathbb{R}^d) \\ &\geq \frac{1}{2} \inf_{|a| \leq C_A \|\bar{B}\| |x-y|} \mathbf{n} \wedge (\delta_a * \mathbf{n})(\mathbb{R}^d) \\ &\geq \frac{1}{2} \inf_{|a| \leq \delta} \mathbf{n} \wedge (\delta_a * \mathbf{n})(\mathbb{R}^d) \\ &=: \frac{1}{2} \gamma(\delta) > 0. \end{aligned} \tag{2.15}$$

We will now estimate $\mathbb{P}(T^S > k)$. Let V_i , $i \geq 1$, be independent symmetric random variables on \mathbb{R} , whose distributions are given by

$$\mathbb{P}(V_i = z) = \begin{cases} \frac{1}{2}(1 - p_i), & \text{if } z = -|e^{tA}(x - y)|; \\ \frac{1}{2}(1 - p_i), & \text{if } z = |e^{tA}(x - y)|; \\ p_i, & \text{if } z = 0. \end{cases}$$

Set $Z_k := \sum_{i=1}^k V_i$. We have seen earlier that

$$T^S = \inf\{k \geq 1 : Z_k = |e^{tA}(x - y)|\}.$$

For any $k \geq 1$, let

$$\eta = \eta(k) := \#\{i : i \leq k \text{ and } V_i \neq 0\}$$

and set $\tilde{Z}_k := \sum_{i=1}^k \tilde{V}_i$, where \tilde{V}_i denotes the i th V_j such that $V_j \neq 0$. Then, \tilde{Z}_k is a symmetric random walk with iid steps which are either $-|e^{tA}(x - y)|$ or $|e^{tA}(x - y)|$ with probability $1/2$. Define

$$T^{\tilde{Z}} := \inf\{k \geq 1 : \tilde{Z}_k = |e^{tA}(x - y)|\}.$$

By (2.15),

$$\begin{aligned} \mathbb{P}(T^S > k) &= \mathbb{P}\left(T^S > k, \eta \geq \frac{1}{2} \gamma(\delta)k\right) + \mathbb{P}\left(T^S > k, \eta \leq \frac{1}{2} \gamma(\delta)k\right) \\ (2.16) \quad &\leq \mathbb{P}\left(T^{\tilde{Z}} > \frac{1}{2} \gamma(\delta)k\right) + \mathbb{P}\left(\eta \leq \frac{1}{2} \sum_{i=1}^k (1 - p_i)\right) \\ &\leq \mathbb{P}\left(T^{\tilde{Z}} > \frac{1}{2} \gamma(\delta)k\right) + \mathbb{P}\left(\left|\eta - \sum_{i=1}^k (1 - p_i)\right| \geq \frac{1}{2} \sum_{i=1}^k (1 - p_i)\right). \end{aligned}$$

Note that

$$\eta = \eta(k) = \sum_{i=1}^k \zeta_i,$$

where $\zeta_i = \mathbf{1}_{\{V_i \neq 0\}}$, $1 \leq i \leq k$, are independent random variables with $\mathbb{P}(\zeta_i = 0) = p_i$ and $\mathbb{P}(\zeta_i = 1) = 1 - p_i$. Chebyshev's inequality shows that

$$\begin{aligned} \mathbb{P}\left(\left|\eta - \sum_{i=1}^k (1 - p_i)\right| \geq \frac{1}{2} \sum_{i=1}^k (1 - p_i)\right) &\leq \frac{4\text{Var}(\eta)}{\left(\sum_{i=1}^k (1 - p_i)\right)^2} \\ (2.17) \quad &= \frac{4 \sum_{i=1}^k p_i(1 - p_i)}{\left(\sum_{i=1}^k (1 - p_i)\right)^2} \\ &\leq \frac{4(1 - \gamma(\delta)) \sum_{i=1}^k (1 - p_i)}{\left(\sum_{i=1}^k (1 - p_i)\right)^2} \\ &\leq \frac{4(1 - \gamma(\delta))}{\gamma(\delta)k}. \end{aligned}$$

For the second and the last inequality we have used (2.15).

On the other hand, by Lemma 2.3 below,

$$\begin{aligned} \mathbb{P}\left(T^{\tilde{Z}} > \frac{\gamma(\delta)k}{2}\right) &= \mathbb{P}\left(\max_{i \leq \left[\frac{\gamma(\delta)k}{2}\right]} \tilde{Z}_i < |e^{tA}(x-y)|\right) \\ &\leq 2 \mathbb{P}\left(0 \leq \tilde{Z}_{\left[\frac{\gamma(\delta)k}{2}\right]} \leq |e^{tA}(x-y)|\right). \end{aligned}$$

From the construction above, we know that $(\tilde{Z}_k)_{k \geq 1}$ is a symmetric random walk with iid steps with values $\pm|e^{tA}(x-y)|$. Using the central limit theorem we find for sufficiently large values of $k \geq k_0$ and some constant $C = C(k_0)$

$$\begin{aligned} \mathbb{P}\left(T^S > \frac{1}{2} \gamma(\delta)k\right) &= 2 \mathbb{P}\left(0 \leq \frac{Z_k}{|e^{tA}(x-y)| \sqrt{\left[\frac{\gamma(\delta)k}{2}\right]}} \leq \left[\frac{\gamma(\delta)k}{2}\right]^{-1/2}\right) \\ (2.18) \quad &\leq \frac{C}{\sqrt{2\pi}} \int_0^{\left[\frac{\gamma(\delta)k}{2}\right]^{-1/2}} e^{-u^2/2} du \\ &\leq \frac{C_{\gamma(\delta)}}{\sqrt{k}}. \end{aligned}$$

Combining (2.16), (2.17) and (2.18) gives for all $x, y \in \mathbb{R}^n$ with $|x-y| \leq \delta(C_A \|\bar{B}\|)^{-1}$, $t \geq t_1 + \dots + t_k$ and $k \geq k_0$ that

$$\mathbb{P}(T^S > k) \leq \frac{C_{\gamma(\delta)}}{\sqrt{k}} + \frac{4(1-\gamma(\delta))}{\gamma(\delta)k}.$$

Finally, (2.14) yields for all $x, y \in \mathbb{R}^n$ with $|x-y| \leq \delta(C_A \|\bar{B}\|)^{-1}$, $t \geq t_1 + \dots + t_k$ and $k \geq 1$, that

$$(2.19) \quad \|\delta_{e^{tA}(x-y)} * \mu_{t_1, \dots, t_k} - \mu_{t_1, \dots, t_k}\|_{\text{Var}} \leq \frac{C_{1, \delta, n}}{\sqrt{k}}.$$

Step 6. If we combine (2.12), (2.13) and (2.19), we obtain that for all $x, y \in \mathbb{R}^n$ with $|x - y| \leq \delta(C_A \|\bar{B}\|)^{-1}$,

$$\begin{aligned}
& \|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{Var}} \\
& \leq 2e^{-C_\varepsilon t} + C_{1,\delta,n} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} \int \cdots \int_{t_1 + \cdots + t_k \leq t < t_1 + \cdots + t_{k+1}} C_\varepsilon^{k+1} e^{-C_\varepsilon(t_1 + \cdots + t_{k+1})} dt_1 \cdots dt_{k+1} \\
& \leq 2e^{-C_\varepsilon t} + C_{1,\delta,n} e^{-C_\varepsilon t} \sum_{k=1}^{\infty} \frac{C_\varepsilon^{k+1}}{\sqrt{k}} \int \cdots \int_{t_1 + \cdots + t_k \leq t} dt_1 \cdots dt_k \\
(2.20) \quad & \leq 2e^{-C_\varepsilon t} + C_{1,\delta,n} C_\varepsilon \sum_{k=1}^{\infty} \frac{C_\varepsilon^k t^k}{\sqrt{k} k!} e^{-C_\varepsilon t} \\
& \leq 2e^{-C_\varepsilon t} + \frac{\sqrt{2} C_{1,\delta,n} C_\varepsilon (1 - e^{-C_\varepsilon t})}{\sqrt{C_\varepsilon t}} \\
& \leq \frac{C_{2,\varepsilon,\delta,n}}{\sqrt{t}},
\end{aligned}$$

where the penultimate inequality follows as in [13, Proposition 2.2].

For any $x, y \in \mathbb{R}^n$, set $k = \left\lceil \frac{C_A \|\bar{B}\| |x - y|}{\delta} \right\rceil + 1$. Pick $x_0, x_1, \dots, x_k \in \mathbb{R}^n$ such that $x_0 = x$, $x_k = y$ and $|x_i - x_{i-1}| \leq \delta(C_A \|\bar{B}\|)^{-1}$ for $1 \leq i \leq k$. By (2.20),

$$\begin{aligned}
\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{Var}} & \leq \sum_{i=1}^k \|P_t(x_i, \cdot) - P_t(x_{i-1}, \cdot)\|_{\text{Var}} \\
& \leq \frac{C_{\varepsilon,\delta,n,A,B}(1 + |x - y|)}{\sqrt{t}},
\end{aligned}$$

which finishes the proof of (1.4). \square

The following two lemmas have been used in the proof of Theorem 1.1 above. For the sake of completeness we include their proofs.

Lemma 2.1. *Let $B \in \mathbb{R}^{n \times d}$ and $(Z_t)_{t \geq 0}$ be a d -dimensional Lévy process with characteristic exponent Φ as in (1.2). Then, $(Z_t^B)_{t \geq 0} := (BZ_t)_{t \geq 0}$ is a Lévy process on (a subspace of) \mathbb{R}^n , and the corresponding characteristic exponent is*

$$\mathbb{R}^n \ni \xi \mapsto \Phi_B(\xi) := \Phi(B^\top \xi).$$

The Lévy triplet (Q_B, b_B, ν_B) of $(Z_t^B)_{t \geq 0}$ is given by $Q_B = BQB^\top$, $\nu_B(C) = \nu\{y : By \in C\}$ and

$$b_B = Bb + \int_{x \neq 0} Bx \left(\mathbb{1}_{\{z \in \mathbb{R}^d : |z| \leq 1\}}(Bx) - \mathbb{1}_{\{z \in \mathbb{R}^d : |z| \leq 1\}}(x) \right) \nu(dx).$$

Proof. For all $\xi \in \mathbb{R}^n$ and $t \geq 0$, we have

$$\mathbb{E}(e^{i\langle \xi, Z_t^B \rangle}) = \mathbb{E}(e^{i\langle \xi, BZ_t \rangle}) = \mathbb{E}(e^{i\langle B^\top \xi, Z_t \rangle}) = e^{-t\Phi(B^\top \xi)}.$$

The assertion follows from (1.2) and some straightforward calculations. \square

Lemma 2.2. *Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times d}$ and $(Z_t)_{t \geq 0}$ be a d -dimensional Lévy process with the characteristic exponent Φ as in (1.2). For all $t > 0$ the random variables $\int_0^t e^{(t-s)A} B dZ_s$ and $\int_0^t e^{sA} B dZ_s$ have the same probability distribution. Furthermore, both random variables are infinitely divisible, and the characteristic exponent (log-characteristic function) is given by*

$$\mathbb{R}^n \ni \xi \mapsto \Phi_t(\xi) := \int_0^t \Phi(B^\top e^{sA^\top} \xi) ds.$$

Proof. We first assume that $n = d$ and $B = \text{id}_{\mathbb{R}^d}$. For any $t > 0$, we can use Lemma 2.1 and follow the proof of [10, (17.3)] to deduce

$$\mathbb{E} \left[\exp \left(i \left\langle \xi, \int_0^t e^{(t-s)A} dZ_s \right\rangle \right) \right] = \exp \left[- \int_0^t \Phi(e^{(t-s)A^\top} \xi) ds \right]$$

for all $\xi \in \mathbb{R}^d$. Similarly, for every $\xi \in \mathbb{R}^d$,

$$\mathbb{E} \left[\exp \left(i \left\langle \xi, \int_0^t e^{sA} dZ_s \right\rangle \right) \right] = \exp \left[- \int_0^t \Phi(e^{sA^\top} \xi) ds \right].$$

Since

$$\exp \left[- \int_0^t \Phi(e^{(t-s)A^\top} \xi) ds \right] = \exp \left[- \int_0^t \Phi(e^{sA^\top} \xi) ds \right],$$

it follows that $\int_0^t e^{(t-s)A} B dZ_s$ and $\int_0^t e^{sA} B dZ_s$ have the same law.

Now replace in the preceding calculations A with $\frac{1}{k} A$, $k \geq 1$, and set $Y_k := \int_0^t e^{s \frac{1}{k} A} dZ_s$. Denote by $Y_k^{(j)}$, $1 \leq j \leq k$, independent copies of Y_k . It is straightforward to see that $\sum_{j=1}^k Y_k^{(j)}$ and $\int_0^t e^{sA} dZ_s$ have the same law. This proves the infinite divisibility.

If $n \neq d$, we consider, as in Lemma 2.1, the Lévy process $(Z_t^B)_{t \geq 0} := (BZ_t)_{t \geq 0}$ on (a subspace of) \mathbb{R}^n . Then, for any $\xi \in \mathbb{R}^n$,

$$\mathbb{E} \left[\exp \left(i \left\langle \xi, \int_0^t e^{(t-s)A} B dZ_s \right\rangle \right) \right] = \mathbb{E} \left[\exp \left(i \left\langle \xi, \int_0^t e^{(t-s)A} dZ_s^B \right\rangle \right) \right],$$

and the claim follows from the first part of our proof. \square

The following result presents the upper estimate for the distribution of the maximum of a symmetric random walk, by using the reflection principle. Since we could not find a precise reference in the literature, we include the complete proof for the readers' convenience.

Lemma 2.3. *Consider a random walk $(S_i)_{i \geq 1}$ on \mathbb{Z} with iid steps, which attain the values -1 , 1 and 0 with probabilities $(1-r)/2$, $(1-r)/2$ and r ($0 \leq r < 1$), respectively. Then for any positive integers a and k , we have*

$$(2.21) \quad 2\mathbb{P}(S_k > a) \leq \mathbb{P} \left(\max_{1 \leq i \leq k} S_i \geq a \right) \leq 2\mathbb{P}(S_k \geq a)$$

and

$$2\mathbb{P}(0 < S_k < a) \leq \mathbb{P} \left(\max_{1 \leq i \leq k} S_i < a \right) \leq 2\mathbb{P}(0 \leq S_k \leq a).$$

Proof. Fix any positive integer a and define $\tau := \tau_a := \inf\{i \geq 1 : S_i = a\}$. Since the random walk has iid steps, it is obvious that $(S_{i+\tau} - S_\tau)_{i \geq 0}$ and $(S_i)_{i \geq 0}$ are independent random walks having the same law. Observing that $S_\tau = a$ and $\{\max_{i \leq k} S_i \geq a\} = \{\tau \leq k\}$ we find, therefore,

$$\begin{aligned} \mathbb{P}\left(\max_{i \leq k} S_i \geq a\right) &= \mathbb{P}\left(\max_{i \leq k} S_i \geq a, S_k \geq a\right) + \mathbb{P}\left(\max_{i \leq k} S_i \geq a, S_k < a\right) \\ &= \mathbb{P}(S_k \geq a) + \mathbb{P}(\tau \leq k, S_k < S_\tau) \\ &= \mathbb{P}(S_k \geq a) + \mathbb{P}(\tau \leq k, S_k > S_\tau) \\ &= \mathbb{P}(S_k \geq a) + \mathbb{P}(S_k > a). \end{aligned}$$

From this we conclude that

$$2\mathbb{P}(S_k \geq a) \geq \mathbb{P}\left(\max_{i \leq k} S_i \geq a\right) \geq 2\mathbb{P}(S_k > a).$$

Since $\mathbb{P}(S_k \geq 0) = \mathbb{P}(S_k \leq 0) \geq 1/2$, we see

$$\begin{aligned} \mathbb{P}\left(\max_{i \leq k} S_i < a\right) &= 1 - \mathbb{P}\left(\max_{i \leq k} S_i \geq a\right) \\ &\leq 1 - 2\mathbb{P}(S_k > a) \\ &\leq 2(\mathbb{P}(S_k \geq 0) - \mathbb{P}(S_k > a)) \\ &= 2\mathbb{P}(0 \leq S_k \leq a); \end{aligned}$$

the other inequality follows similarly if we use $\mathbb{P}(S_k > 0) = \mathbb{P}(S_k < 0) \leq 1/2$. \square

Next, we turn to the proof of Theorem 1.7.

Proof of Theorem 1.7. Step 1. As in the proof of Lemma 2.2 we may, without loss of generality, assume that $n = d$ and $B = \text{id}_{\mathbb{R}^d}$. For $t > 0$, denote by μ_t the law of $X_t^0 := \int_0^t e^{(t-s)A} dZ_s$. According to Lemma 2.2, the law μ_t is an infinitely divisible probability distribution, and the characteristic exponent of μ_t is given by

$$\Phi_t(\xi) := \int_0^t \Phi(e^{sA^\top} \xi) ds.$$

Since the driving Lévy process $(Z_t)_{t \geq 0}$ has no Gaussian part, the Lévy triplet $(0, b_t, \nu_t)$ of Φ_t is given by, cf. [9, Theorem 3.1],

$$\begin{aligned} \nu_t(C) &= \int_0^t \nu(e^{-sA} C) ds, \quad C \in \mathcal{B}(\mathbb{R}^d \setminus \{0\}), \\ b_t &= \int_0^t e^{sA} b ds + \int_{z \neq 0} \int_0^t e^{sA} z \left(\mathbb{1}_{\{|z| \leq 1\}}(e^{sA} z) - \mathbb{1}_{\{|z| \leq 1\}}(z) \right) ds \nu(dz). \end{aligned}$$

For every $r > 0$, let $\{\mu_t^r, t \geq 0\}$ be the family of infinitely divisible probability measures on \mathbb{R}^d whose Fourier transform is of the form $\widehat{\mu}_t^r(\xi) = \exp(-\Phi_{t,r}(\xi))$, where

$$\Phi_{t,r}(\xi) = \int_{|z| \leq r} (1 - e^{i\langle \xi, z \rangle} + i\langle \xi, z \rangle) \nu_t(dz)$$

with ν_t as above.

Set $h(t) := 1/\varphi_t^{-1}(1)$. Following the proof of [12, Propostion 2.2], the conditions (1.6) and (1.7) ensure that there exists $t_1 > 0$ such that for all $t \geq t_1$, the measure $\mu_t^{h(t)}$ has a density $p_t^{h(t)} \in C_b^{n+2}(\mathbb{R}^d)$; moreover,

$$(2.22) \quad |\nabla p_t^{h(t)}(y)| \leq c(n, \Phi) h(t)^{-(n+1)} (1 + h(t)^{-1}|y|)^{-(n+1)}$$

holds for all $y \in \mathbb{R}^d$.

Step 2. For $r > 0$ and $\xi \in \mathbb{R}^d$, define

$$\Psi_{t,r}(\xi) := \Phi_t(\xi) - \Phi_{t,r}(\xi) = \int_{|z|>r} (1 - e^{i\langle \xi, z \rangle}) \nu_t(dz) - i \left\langle \xi, \int_{1<|z|\leq r} z \nu_t(dz) - b_t \right\rangle.$$

Since $\Psi_{t,r}$ is given by a Lévy-Khintchine formula, it is the characteristic exponent of some d -dimensional infinitely divisible random variable. Let $\{\pi_t^r, t \geq 0\}$ be the family of infinitely divisible measures whose Fourier transforms are of the form $\widehat{\pi}_t^r(\xi) = \exp(-\Psi_{t,r}(\xi))$. Clearly, $\mu_t = \mu_t^r * \pi_t^r$ for all $t, r > 0$.

Let $P_t(x, \cdot)$ and P_t be the transition function and the transition semigroup of the Ornstein-Uhlenbeck process $\{X_t^x\}_{t \geq 0}$ given by (1.1). For all $f \in B_b(\mathbb{R}^d)$ we have

$$\begin{aligned} P_t f(x) &= \int f(e^{tA}x + z) \mu_t(dz) \\ &= \int f(e^{tA}x + z) \mu_t^r * \pi_t^r(dz) \\ &= \iint f(e^{tA}x + z_1 + z_2) \pi_t^r(dz_1) \mu_t^r(dz_2). \end{aligned}$$

Taking $r = h(t)$ we get, using the conclusions of step 1, that for all $t \geq t_1$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} P_t f(x) &= \int p_t^{h(t)}(z_2) dz_2 \int f(e^{tA}x + z_1 + z_2) \pi_t^{h(t)}(dz_1) \\ &= \int p_t^{h(t)}(z_2 - e^{tA}x) dz_2 \int f(z_1 + z_2) \pi_t^{h(t)}(dz_1). \end{aligned}$$

If $\|f\|_\infty \leq 1$, then

$$\left\| \int f(z_1 + \cdot) \pi_t^{h(t)}(dz_1) \right\|_\infty \leq \|f\|_\infty \pi_t^{h(t)}(\mathbb{R}^d) \leq 1.$$

Step 3. For all $x, y \in \mathbb{R}^d$,

$$\begin{aligned}
& \|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} \\
&= \sup_{\|f\|_\infty \leq 1} |P_t f(x) - P_t f(y)| \\
&= \sup_{\|f\|_\infty \leq 1} \left| \int p_t^{h(t)}(z_2 - e^{tA}x) dz_2 \int f(z_1 + z_2) \pi_t^{h(t)}(dz_1) \right. \\
&\quad \left. - \int p_t^{h(t)}(z_2 - e^{tA}y) dz_2 \int f(z_1 + z_2) \pi_t^{h(t)}(dz_1) \right| \\
(2.23) \quad & \leq \sup_{\|g\|_\infty \leq 1} \left| \int g(z) p_t^{h(t)}(z - e^{tA}x) dz - \int g(z) p_t^{h(t)}(z - e^{tA}y) dz \right| \\
&= \sup_{\|g\|_\infty \leq 1} \left| \int g(z) \left(p_t^{h(t)}(z - e^{tA}x) - p_t^{h(t)}(z - e^{tA}y) \right) dz \right| \\
&= \int \left| p_t^{h(t)}(z - e^{tA}x) - p_t^{h(t)}(z - e^{tA}y) \right| dz.
\end{aligned}$$

With the argument used in the proof of [12, Theorem 3.1], (1.8) follows from (2.22) and (2.23).

Step 4. By assumption (1.9),

$$\varphi_\infty(\rho) := \sup_{|\xi| \leq \rho} \int_0^\infty \text{Re} \Phi(B^\top e^{sA^\top} \xi) ds$$

is finite on $(0, \infty)$; in particular, $\varphi_\infty^{-1}(1) \in (0, \infty]$. On the other hand, for any $t \geq t_0$, according to (1.6),

$$\begin{aligned}
& \int \exp \left(- \int_0^t \text{Re} \Phi(B^\top e^{sA^\top} \xi) ds \right) |\xi|^{n+2} d\xi \\
& \leq \int \exp \left(- \int_0^{t_0} \text{Re} \Phi(B^\top e^{sA^\top} \xi) ds \right) |\xi|^{n+2} d\xi \\
& =: C(t_0) < \infty.
\end{aligned}$$

Since the function $t \mapsto \varphi_t^{-1}(1)$ is decreasing on $(0, \infty]$, (1.7) holds. This finishes the proof. \square

3. APPENDIX

3.1. Gradient Estimates for Ornstein-Uhlenbeck Processes. Motivated by [12, Theorem 1.3], we have the following results for gradient estimates of an Ornstein-Uhlenbeck process. This is the counterpart of Theorem 1.7. For $t, \rho > 0$, define

$$\varphi(\rho) := \sup_{|\xi| \leq \rho} \text{Re} \Phi(B^\top \xi) \quad \text{and} \quad \varphi_t(\rho) := \sup_{|\xi| \leq \rho} \int_0^t \text{Re} \Phi(B^\top e^{sA^\top} \xi) ds,$$

where Φ is the characteristic exponent of the driving Lévy process $(Z_t)_{t \geq 0}$ from (1.1).

Theorem 3.1. *Let $P_t(x, \cdot)$ be the transition function of the n -dimensional Ornstein-Uhlenbeck process $\{X_t^x\}_{t \geq 0}$ given by (1.1). Assume that*

$$(3.24) \quad \liminf_{|\xi| \rightarrow \infty} \frac{\operatorname{Re} \Phi(B^\top \xi)}{\log(1 + |\xi|)} = \infty.$$

If for any $C > 0$,

$$(3.25) \quad \int \exp[-Ct \operatorname{Re} \Phi(B^\top \xi)] |\xi|^{n+2} d\xi = O\left(\varphi^{-1}\left(\frac{1}{t}\right)^{2n+2}\right) \quad \text{as } t \rightarrow 0,$$

then there exists $c > 0$ such that for all $t > 0$ and $f \in B_b(\mathbb{R}^n)$,

$$(3.26) \quad \|\nabla P_t f\|_\infty \leq c \|f\|_\infty \varphi^{-1}\left(\frac{1}{t \wedge 1}\right).$$

If, in addition,

$$\xi \mapsto \int_0^\infty \operatorname{Re} \Phi(B^\top e^{sA^\top} \xi) ds \quad \text{is locally bounded,}$$

then there exist $t_1, c > 0$ such that for $t \geq t_1$ and $f \in B_b(\mathbb{R}^n)$,

$$(3.27) \quad \|\nabla P_t f\|_\infty \leq c \|f\|_\infty \left[\|e^{tA}\| \varphi_t^{-1}(1) \right],$$

where $\|M\| = \sup_{|x| \leq 1} |Mx|$ denotes the norm of the matrix of M .

To illustrate the power of Theorem 3.1, we consider

Example 3.2. Let μ be a finite nonnegative measure on the unit sphere $\mathbb{S} \subset \mathbb{R}^n$ and assume that μ is nondegenerate in the sense that its support is not contained in any proper linear subspace of \mathbb{R}^n . Let $\alpha \in (0, 2)$, $\beta \in (0, \infty]$ and assume that the Lévy measure ν satisfies

$$\nu(C) \geq \int_0^{r_0} \int_{\mathbb{S}} \mathbb{1}_C(s\theta) s^{-1-\alpha} ds \mu(d\theta) + \int_{r_0}^\infty \int_{\mathbb{S}} \mathbb{1}_C(s\theta) s^{-1-\beta} ds \mu(d\theta)$$

for some constant $r_0 > 0$ and all $C \in \mathcal{B}(\mathbb{R}^n \setminus \{0\})$. Consider the following Ornstein-Uhlenbeck process X_t on \mathbb{R}^n given by

$$dX_t = AX_t dt + dZ_t,$$

where $(Z_t)_{t \geq 0}$ is a Lévy process on \mathbb{R}^n with the Lévy measure ν . By Theorem 3.1 there exists a constant $c > 0$ such that for all $t > 0$ and $f \in B_b(\mathbb{R}^n)$,

$$\|\nabla P_t f\|_\infty \leq c \|f\|_\infty (t \wedge 1)^{-1/\alpha}.$$

Furthermore, if the real parts of all eigenvalues of A are negative, then there exists a constant $c > 0$ such that for all $t > 0$ and $f \in B_b(\mathbb{R}^n)$,

$$\|\nabla P_t f\|_\infty \leq c \|f\|_\infty \frac{\|e^{tA}\|}{(t \wedge 1)^{1/\alpha}}.$$

Recently, F.-Y. Wang [17, Theorem 1.1] has presented explicit gradient estimates for Ornstein-Uhlenbeck processes, by assuming that the corresponding Lévy measure has absolutely continuous (*with respect to Lebesgue measure*) lower bounds. Since lower bounds of Lévy measure in Example 3.2 could be much irregular, Theorem 3.1 is more applicable than [17, Theorem 1.1].

Sketch of the Proof of Theorem 3.1. Assuming the conditions (3.24) and (3.25), we can mimic the proof of [12, Theorem 3.2] to show that there exist $t_1, C > 0$ such that for all $x, y \in \mathbb{R}^n$ and $t \leq t_1$,

$$\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} \leq C |e^{tA}(x - y)| \varphi^{-1}\left(\frac{1}{t}\right).$$

Thus we can apply to find for all $f \in B_b(\mathbb{R}^n)$ with $\|f\|_\infty = 1$,

$$\begin{aligned} |\nabla P_t f(x)| &\leq \limsup_{y \rightarrow x} \frac{|P_t f(x) - P_t f(y)|}{|y - x|} \\ &\leq \limsup_{y \rightarrow x} \frac{\sup_{\|w\|_\infty \leq 1} |P_t w(x) - P_t w(y)|}{|y - x|} \\ (3.28) \quad &\leq \limsup_{y \rightarrow x} \frac{\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}}}{|y - x|} \\ &\leq C \|e^{tA}\| \varphi^{-1}\left(\frac{1}{t}\right) \\ &\leq \left[C \sup_{s \leq t_1} \|e^{sA}\| \right] \varphi^{-1}\left(\frac{1}{t}\right). \end{aligned}$$

Because of the Markov property of the semigroup P_t , the function

$$t \mapsto \sup_{f \in B_b(\mathbb{R}^n), \|f\|_\infty = 1} \|\nabla P_t f\|_\infty$$

is decreasing. Combining this and (3.28) yields (3.26).

The assertion (3.27) follows if we combine the above argument with (1.8): there exist $t_2, C > 0$ such that for all $x, y \in \mathbb{R}^n$ and $t \geq t_2$,

$$\|P_t(x, \cdot) - P_t(y, \cdot)\|_{\text{var}} \leq C |e^{tA}(x - y)| \varphi_t^{-1}(1). \quad \square$$

3.2. Proof of Proposition 1.5.

Proof of Proposition 1.5. Because of (1.5), we can choose a closed subset $F \subset \overline{B(z_0, \varepsilon)}$ such that $0 \notin F$ and

$$\int_F \frac{dz}{\rho_0(z)} < \infty.$$

By the Cauchy-Schwarz inequality, we have

$$\left(\int_F \rho_0(z) dz \right)^{-1} \leq \frac{1}{\text{Leb}(F)^2} \int_F \frac{dz}{\rho_0(z)} < \infty.$$

Hence,

$$K := \int_F \rho_0(z) dz > 0.$$

Since F is a compact set and $0 \notin F$, there exists some $\delta_0 > 0$ such that $0 \notin F + \overline{B(0, \delta_0)}$, where $F + \overline{B(0, \delta_0)} := \{a + b : a \in F, |b| \leq \delta_0\}$. Since ρ_0 is locally integrable, we know that

$$K \leq \int_{F + \overline{B(0, \delta_0)}} \rho_0(z) dz < \infty.$$

The remainder of the proof is now similar to the argument which shows that the shift $x \mapsto \|f(\cdot - x) - f\|_{L^1}$, $f \in L^1(\mathbb{R}^d, \text{Leb})$, is continuous, see e.g. [14, Lemma 6.3.5] or [11, Theorem 14.8]: choose $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\text{supp } \chi \subset F + \overline{B(0, \delta_0)}$ and

$$\int_{F + \overline{B(0, \delta_0)}} |\rho_0(z) - \chi(z)| dz \leq \frac{K}{4}.$$

Therefore, for any $x \in \mathbb{R}^d$ with $|x| \leq \delta_0$, we obtain

$$\begin{aligned} & \int_F |\rho_0(z) - \rho_0(z - x)| dz \\ & \leq \int_F |\rho_0(z) - \chi(z)| dz + \int_F |\chi(z) - \chi(z - x)| dz + \int_F |\rho_0(z - x) - \chi(z - x)| dz \\ & = \int_F |\rho_0(z) - \chi(z)| dz + \int_F |\chi(z) - \chi(z - x)| dz + \int_{F+x} |\rho_0(z) - \chi(z)| dz \\ & \leq 2 \int_{F + \overline{B(0, \delta_0)}} |\rho_0(z) - \chi(z)| dz + \int_F |\chi(z) - \chi(z - x)| dz \\ & \leq \frac{K}{2} + \int_F |\chi(z) - \chi(z - x)| dz. \end{aligned}$$

By the dominated convergence theorem we see that

$$x \mapsto \int_F |\chi(z) - \chi(z - x)| dz$$

is continuous on \mathbb{R}^d . Therefore, there exists $0 < \delta \leq \delta_0$ such that

$$\sup_{x \in \mathbb{R}^d, |x| \leq \delta} \int_F |\chi(z) - \chi(z - x)| dz \leq \frac{K}{4}$$

and, in particular,

$$\sup_{x \in \mathbb{R}^d, |x| \leq \delta} \int_F |\rho_0(z) - \rho_0(z - x)| dz \leq \frac{3K}{4}.$$

Using $2(a \wedge b) = a + b - |a - b|$ for all $a, b \geq 0$, we get

$$\begin{aligned}
& \inf_{x \in \mathbb{R}^d, |x| \leq \delta} \int_F (\rho_0(z) \wedge \rho_0(z - x)) \, dz \\
&= \frac{1}{2} \inf_{x \in \mathbb{R}^d, |x| \leq \delta} \left[\int_F (\rho_0(z) + \rho_0(z - x)) \, dz - \int_F |\rho_0(z) - \rho_0(z - x)| \, dz \right] \\
&\geq \frac{1}{2} \int_F \rho_0(z) \, dz - \frac{1}{2} \sup_{x \in \mathbb{R}^d, |x| \leq \delta} \int_F |\rho_0(z) - \rho_0(z - x)| \, dz \\
&\geq \frac{K}{8} > 0.
\end{aligned}$$

This finishes the proof. \square

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